Chapter 2

Expected Utility Theory and Possible States of Zero Probability[*]

ABSTRACT: At least at first glance, Expected Utility Theory tells us to be indifferent between two prospects when they are otherwise the same, except that one gives a better outcome than the other in a possible state of zero probability. But as some have suggested, Expected Utility Theory might be supplemented with dominance to get the verdict that the dominating prospect is better than the dominated one. However, I will show that if Expected Utility Theory is supplemented with dominance in this way, it will violate the Continuity axiom of Expected Utility Theory.

Consider the following principle of rationality:

> **Statewise Dominance:** If the outcome of prospect X is at least as preferred as the outcome of prospect Y in all states, then X is at least as good as Y. Furthermore, if in addition the outcome of X is strictly

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preferred to the outcome of Y in some possible state, then X is strictly better than Y.

Hájek (2014, pp. 556–558) presents a case in which Expected Utility Theory violates Statewise Dominance when the principle is formulated in this way.[1] This dominance violation happens because, although we tend to think of probability zero as meaning impossible, this is not strictly true. Consider the following prospects:

Prospect A A fair coin is tossed an infinite number of times. If the coin lands heads on every toss, then the agent goes to heaven; otherwise, nothing happens.

Prospect B As above, but the agent goes to hell if the coin lands heads on every toss; otherwise, nothing happens.

In this case, A statewise dominates B. However, Expected Utility Theory assigns the same expected utility to both prospects because the probability that the coin lands heads on every toss is zero. Consequently, Expected Utility Theory permits the choice of a statewise-dominated prospect.

To avoid violating Statewise Dominance in this way, Hájek (2014, p. 556) argues that decision-makers should sometimes consider states of probability zero. He argues that there is more to the machinery of decision theory than just Expected

[1]Let EU (X) denote the expected utility of prospect X and let $X \succsim Y$ mean that X is at least as good as Y. Also, let O be the set of possible outcomes, $p_X(o)$ the probability of outcome o in prospect X and $u(o)$ the utility of o. Then, Expected Utility Theory states the following:

Expected Utility Theory: For all prospects X and Y, $X \succsim Y$ if and only if EU(X) $\geq$ EU(Y), where

$$\text{EU}(X) = \sum_{o \in O} p_X(o)u(o).$$

Utility Theory, and he goes on to suggest that Expected Utility Theory can be supplemented with dominance.[2] He argues, following Easwaran (2014), that there is no conflict between dominance and Expected Utility Theory in this case because Expected Utility Theory should not be interpreted as telling us that prospects with tied expected utilities must be treated with indifference. Instead, he argues that it should be interpreted as failing to tell us anything in such cases. So, without conflicting with what Expected Utility Theory tells us, we may choose on some other basis. As Easwaran (2014, p. 14) writes, in cases where the expected utility of a bet is the same as the status quo, some non-numerical feature may serve as a tiebreaker. Hájek (2014, p. 557) suggests that dominance may serve as a tiebreaker between A and B. However, I will show that expected utility theorists cannot use Statewise Dominance to argue that A is better than B, at least if they wish to keep standard axiomatizations of Expected Utility Theory.

1 A violation of Continuity

Let $X \succ Y$ mean that X is better than Y. Also, let XpY be a risky prospect with a p chance of prospect X obtaining and a $1 - p$ chance of prospect Y obtaining. Then, using Statewise Dominance to argue that A is better than B would result in

[2] Hájek (2014, p. 557). Russell (2021, p. 12–13 n. 9) also suggests that a prospect that spares a child from malaria if an ideally sharp dart hits a particular point (and otherwise nothing happens) may be better than the prospect of certainly getting nothing, even though it gives a probability zero of a positive outcome. Russell suggests that what is best may depend on what features of its outcomes are sure, which can come apart from what is almost sure.

a violation of the following axiom of Expected Utility Theory:[3]

Continuity: If $X \succ Y \succ Z$, then there are probabilities p and $q \in (0, 1)$ such that $XpZ \succ Y \succ XqZ$.

To see why using Statewise Dominance would result in a violation of Continuity, consider the following case (see table 1).

Prospect A^* A fair coin is tossed an infinite number of times. The agent gets \$10 if the coin lands heads on every toss; otherwise, nothing happens.

Prospect B^* As above, but the agent gets \$1 if the coin lands heads on every toss; otherwise, nothing happens.

Prospect C Certainly gives $-$\$10 (the agent loses \$10).

By Statewise Dominance, A^* is better than B^*, which is better than C.

Next, consider the following mixed prospect:

Prospect A^*pC Gives A^* with probability p and C with probability $1 - p$.

In this case, A^*pC is worse than B^* for all probabilities $p \in (0, 1)$. This is so because A^*pC gives a probability p of nothing and a (non-zero) probability $1 - p$ of losing \$10, while B^* gives a probability one of nothing. Suppose the utility of money equals the monetary amount. Consequently, the expected utility of A^*pC

[3]Jensen (1967, p. 174). Note that strictly speaking Statewise Dominance is undefined in the framework of decision theory under risk, as this notion pertains to decision theory under uncertainty, where there is an explicit underlying state space.

is $EU(A^*pC) < 0$, and the expected utility of B^* is $EU(B^*) = 0$.[4] So, now we have that A^* is better than B^*, which is better than C, but A^*pC is worse than B^* for all probabilities $p \in (0,1)$—which is a violation of Continuity.[5]

Table 1

A Violation of Continuity

Probability	0	p	$1-p$
A^*pC	\$10	\$0	$-$\$10
B^*	\$1	\$0	\$0

2 Conclusion

To conclude, standard axiomatizations of Expected Utility Theory are incompatible with using Statewise Dominance in cases that involve possible states of probability zero because doing so would result in a violation of the Continuity axiom.[6]

[4] $EU(A^*pC) = (1-p) \times (-10) = -10 + 10p < 0$ for all probabilities $p \in (0,1)$.

[5] Some might insist that the probability that the coin lands heads on every toss is not zero but infinitesimal. See for example Lewis (1980, p. 270) and Hájek (2014, p. 556 n. 19). It is unclear how infinitesimal probabilities figure in the decision-making process. If they can only serve as tiebreakers in cases where the prospects are otherwise equally preferable, then A^*pC is still worse than B^* for all probabilities $p \in (0,1)$. So, using infinitesimal probabilities does not help avoid violating Continuity. On the other hand, utilities associated with infinitesimal probabilities might do something more than merely serve as tiebreakers. But it is unclear what their role would be. Any positive or negative contributions to utility would depart from Expected Utility Theory. Note that those who appeal to infinitesimal probabilities might weaken Continuity to accommodate non-Archimedean probabilities. See for example Hammond (1994). See Williamson (2007) for an argument against the appeal to infinitesimals.

[6] Note that if we define Statewise Dominance not in terms of possible states but in terms of states that have non-zero and non-infinitesimal probabilities (as is typically done), then Expected Utility Theory does not violate Statewise Dominance:

> **Statewise Dominance (Non-Zero and Non-Infinitesimal Probabilities:)** If the outcome of prospect X is at least as preferred as the outcome of prospect Y in all

References

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states with non-zero and non-infinitesimal probability, then X is at least as good as Y. Furthermore, if in addition the outcome of X is strictly preferred to the outcome of Y in some state with a non-zero and non-infinitesimal probability, then X is strictly better than Y.

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